# Persistence in a dynamic moral hazard game

## *Online Appendix*

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## B Additional Material for Theorem 1

This section proves Lemmas 2 and 3. Define the second order differential equation

$$U'' = f(X, U, U'), \tag{29}$$

where $f : \mathcal{X} \times \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$f(X, U, U') \equiv \frac{2r}{\sigma^*(X, U')^2}(U - \psi(X, U')) \tag{30}$$

and $\psi(X, z) \equiv g^*(X, z) + \frac{z}{r}\mu^*(X, z)$ is the value of the large player's incentive constraint at the sequentially rational action profile for incentive weight $z/r$. Note $f$ is continuous on $\text{int}(\mathcal{X})$. Eq. (29) is equivalent to the optimality equation Eq. (7). Lemmas 2 and 3 establish that Eq. (29) has a solution for the case of an unbounded and bounded state space, respectively.

The proof of Lemma 2 relies on Theorem 5.6 from De Coster and Patrick (2006), which is reproduced below.[1]

**Theorem 5** (Coster Habets (2006))**.** *Let $\underline{\alpha}, \overline{\alpha} \in \mathcal{C}^2$ be functions such that $\underline{\alpha} \leq \overline{\alpha}$, $D = \{(t, u, v) \in \mathbb{R}^3 | \underline{\alpha}(t) \leq u \leq \overline{\alpha}(t)\}$ and let $f : D \to \mathbb{R}$ be a continuous function. Assume that $\underline{\alpha}$ and $\overline{\alpha}$ are such that for all $t \in \mathbb{R}$,*

$$f(t, \underline{\alpha}(t), \underline{\alpha}'(t)) \leq \underline{\alpha}''(t) \text{ and } \overline{\alpha}''(t) \leq f(t, \overline{\alpha}(t), \overline{\alpha}'(t)).$$

[1] This result is based on Schmitt (1969).

*Assume that for any bounded interval $I$, there exists a positive continuous function $H_I : \mathbb{R}^+ \to \mathbb{R}$ that satisfies the Nagumo condition,*[2]

$$\int_0^\infty \frac{sds}{H_I(s)} = \infty, \tag{31}$$

*and for all $(t, u, v) \in I \times \mathbb{R}^2$ with $\underline{\alpha}(t) \leq u \leq \overline{\alpha}(t)$, $|f(t, u, v)| \leq H_I(|v|)$. Then the equation $u'' = f(t, u, u')$ has at least one solution $u \in \mathcal{C}^2$ such that for all $t \in \mathbb{R}$, $\underline{\alpha}(t) \leq u(t) \leq \overline{\alpha}(t)$.*

**Proof of Lemma 2.** Suppose $\mathcal{X} = \mathbb{R}$. Then Eq. (30) is continuous on $\mathbb{R}^3$. Define $\underline{\alpha} : \mathbb{R} \to \mathbb{R}$ as

$$\underline{\alpha}(X) \equiv \begin{cases} \underline{\alpha}_1 X - c_a & \text{if } X \leq -1 \\ \frac{1}{8}\underline{\alpha}_1 X^4 - \frac{3}{4}\underline{\alpha}_1 X^2 - \frac{3}{8}\underline{\alpha}_1 - c_a & \text{if } X \in (-1, 1) \\ -\underline{\alpha}_1 X - c_a & \text{if } X \geq 1 \end{cases} \tag{32}$$

and $\overline{\alpha} : \mathbb{R} \to \mathbb{R}$ as

$$\overline{\alpha}(X) \equiv \begin{cases} -\overline{\alpha}_1 X + c_b & \text{if } X \leq -1 \\ -\frac{1}{8}\overline{\alpha}_1 X^4 + \frac{3}{4}\overline{\alpha}_1 X^2 + \frac{3}{8}\overline{\alpha}_1 + c_b & \text{if } X \in (-1, 1) \\ \overline{\alpha}_1 X + c_b & \text{if } X \geq 1 \end{cases} \tag{33}$$

for some $\underline{\alpha}_1, \overline{\alpha}_1, c_a, c_b \geq 0$. Note that $\underline{\alpha}, \overline{\alpha} \in \mathcal{C}^2$ and $\underline{\alpha}(X) \leq \overline{\alpha}(X)$ for all $X \in \mathbb{R}$. Functions $\underline{\alpha}(\cdot)$ and $\overline{\alpha}(\cdot)$ are lower and upper solutions to Eq. (29) if there exist $\underline{\alpha}_1, \overline{\alpha}_1, c_a, c_b \geq 0$ such that for all $X \in \mathbb{R}$,

$$\frac{2r}{\sigma(X, \underline{\alpha}'(X))^2} \left(\underline{\alpha}(X) - \psi(X, \underline{\alpha}'(X))\right) \leq \underline{\alpha}''(X) \tag{34}$$

and

$$\overline{\alpha}''(X) \leq \frac{2r}{\sigma(X, \overline{\alpha}'(X))^2} \left(\overline{\alpha}(X) - \psi(X, \overline{\alpha}'(X))\right). \tag{35}$$

By Assumption 2, $\exists k \in [0, r)$ and $c \geq 0$ such that $\mu^*(X, z) \leq kX + c$ for all $X \geq 0$ and $\mu^*(X, z) \geq kX - c$ for all $X \leq 0$.

*Step 1a.* Show that there exist $\underline{\alpha}_1, \overline{\alpha}_1, c_a, c_b \geq 0$ such that $\underline{\alpha}(\cdot)$ and $\overline{\alpha}(\cdot)$ are lower and upper solutions to Eq. (29) when $g$ is unbounded.

First derive a bound on $\psi(X, z)$. By Lipschitz continuity and the fact that $g^*(X, z)$ and $\mu^*(X, z)$ are bounded in $z$, $\exists k_g, k_m \geq 0$ such that $|g^*(X, z) - g^*(0, z)| \leq k_g |X|$ and

[2]The Nagumo condition is a growth condition on the second order differential equation $f$. It plays an important role in demonstrating the existence of a solution to the boundary value problem.

$|\mu^*(X,z) - \mu^*(0,z)| \leq k_m |X|$ for all $(X,z)$. Therefore, $\exists \underline{g}_1, \underline{g}_2, \overline{g}_1, \overline{g}_2 \geq 0$, $\underline{\mu}_1, \overline{\mu}_2 \in [0,r)$, $\underline{\mu}_2, \overline{\mu}_1 > 0$ and $\overline{\gamma}, \underline{\gamma}, \overline{m}, \underline{m} \in \mathbb{R}$ such that:

$$\begin{cases} \underline{g}_1 X + \underline{\gamma} \\ -\underline{g}_2 X + \underline{\gamma} \end{cases} \leq \; g^*(X,z) \leq \begin{cases} -\overline{g}_1 X + \overline{\gamma} & \text{if } X < 0 \\ \overline{g}_2 X + \overline{\gamma} & \text{if } X \geq 0 \end{cases}$$

$$\begin{cases} \underline{\mu}_1 X + \underline{m} \\ -\underline{\mu}_2 X + \underline{m} \end{cases} \leq \; \mu^*(X,z) \leq \begin{cases} -\overline{\mu}_1 X + \overline{m} & \text{if } X < 0 \\ \overline{\mu}_2 X + \overline{m} & \text{if } X \geq 0 \end{cases}$$

and

$$\begin{cases} \left(\underline{g}_1 - \frac{\overline{\mu}_1}{r} z\right) X + \underline{\gamma} + \frac{\overline{m}}{r} z \\ \left(-\underline{g}_2 + \frac{\overline{\mu}_2}{r} z\right) X + \underline{\gamma} + \frac{\overline{m}}{r} z \\ \left(\underline{g}_1 + \frac{\underline{\mu}_1}{r} z\right) X + \underline{\gamma} + \frac{\underline{m}}{r} z \\ -\left(\underline{g}_2 + \frac{\underline{\mu}_2}{r} z\right) X + \underline{\gamma} + \frac{\underline{m}}{r} z \end{cases} \leq \psi(X,z) \leq \begin{cases} \left(-\overline{g}_1 + \frac{\underline{\mu}_1}{r} z\right) X + \overline{\gamma} + \frac{\underline{m}}{r} z & \text{if } X < 0, z \leq 0 \\ \left(\overline{g}_2 - \frac{\underline{\mu}_2}{r} z\right) X + \overline{\gamma} + \frac{\underline{m}}{r} z & \text{if } X \geq 0, z \leq 0 \\ -\left(\overline{g}_1 + \frac{\overline{\mu}_1}{r} z\right) X + \overline{\gamma} + \frac{\overline{m}}{r} z & \text{if } X < 0, z \geq 0 \\ \left(\overline{g}_2 + \frac{\overline{\mu}_2}{r} z\right) X + \overline{\gamma} + \frac{\overline{m}}{r} z & \text{if } X \geq 0, z \geq 0 \end{cases}$$

This provides a bound on $\psi(X,z)$.

Next find conditions on $(\underline{\alpha}_1, \overline{\alpha}_1, c_a, c_b)$ such that $\underline{\alpha}(\cdot)$ and $\overline{\alpha}(\cdot)$ are lower and upper solutions to Eq. (29) when $X \leq -1$. From Eqs. (32) and (33), $\underline{\alpha}''(X) = \overline{\alpha}''(X) = 0$, $\underline{\alpha}'(X) = \underline{\alpha}_1$ and $\overline{\alpha}'(X) = -\overline{\alpha}_1$ when $X \leq -1$. Substituting this into Eqs. (34) and (35), this corresponds to finding $\underline{\alpha}_1, \overline{\alpha}_1, c_a, c_b \geq 0$ such that $\psi(X, \underline{\alpha}_1) \geq \underline{\alpha}_1 X - c_a$ and $\psi(X, -\overline{\alpha}_1) \leq -\overline{\alpha}_1 X + c_b$. From the bound on $\psi(X,z)$,

$$\begin{aligned} \psi(X, \underline{\alpha}_1) &\geq \left(\underline{g}_1 + \frac{\underline{\mu}_1}{r}\underline{\alpha}_1\right) X + \underline{\gamma} + \frac{\underline{m}}{r}\underline{\alpha}_1 \\ \psi(X, -\overline{\alpha}_1) &\leq -\left(\overline{g}_1 + \frac{\underline{\mu}_1}{r}\overline{\alpha}_1\right) X + \overline{\gamma} - \frac{\underline{m}}{r}\overline{\alpha}_1. \end{aligned}$$

Therefore, when $X \leq -1$, this holds when $\underline{\alpha}_1 \geq r\underline{g}_1/(r - \underline{\mu}_1)$, $c_a \geq c_{a1} \equiv -\underline{\gamma} - \underline{m}\ \underline{\alpha}_1/r$, $\overline{\alpha}_1 \geq r\overline{g}_1/(r - \underline{\mu}_1)$ and $c_b \geq c_{b1} \equiv \overline{\gamma} - \underline{m}\overline{\alpha}_1/r$.

Next find conditions on $(\underline{\alpha}_1, \overline{\alpha}_1, c_a, c_b)$ such that $\underline{\alpha}(\cdot)$ and $\overline{\alpha}(\cdot)$ are lower and upper solutions to Eq. (29) when $X \geq 1$. From Eqs. (32) and (33), $\underline{\alpha}''(X) = \overline{\alpha}''(X) = 0$, $\underline{\alpha}'(X) = -\underline{\alpha}_1$ and $\overline{\alpha}'(X) = \overline{\alpha}_1$ when $X \geq 1$. Substituting this into Eqs. (34) and (35), this corresponds to finding $\underline{\alpha}_1, \overline{\alpha}_1, c_a, c_b \geq 0$ such that $\psi(X, -\underline{\alpha}_1) \geq -\underline{\alpha}_1 X - c_a$ and $\psi(X, \overline{\alpha}_1) \leq \overline{\alpha}_1 X + c_b$. From the bound on $\psi(X,z)$,

$$\begin{aligned}\psi(X,-\underline{\alpha}_1) &\geq -\left(\underline{g}_2+\frac{\overline{\mu}_2}{r}\underline{\alpha}_1\right)X+\underline{\gamma}-\frac{\overline{m}}{r}\underline{\alpha}_1\\ \psi(X,\overline{\alpha}_1) &\leq \left(\overline{g}_2+\frac{\overline{\mu}_2}{r}\overline{\alpha}_1\right)X+\overline{\gamma}+\frac{\overline{m}}{r}\overline{\alpha}_1.\end{aligned}$$

Therefore, when $X\geq 1$, this holds when $\underline{\alpha}_1\geq r\underline{g}_2/(r-\overline{\mu}_2)$, $c_a\geq c_{a2}\equiv -\underline{\gamma}+\overline{m}\underline{\alpha}_1/r$, $\overline{\alpha}_1\geq r\overline{g}_2/(r-\overline{\mu}_2)$, and $c_b\geq c_{b2}\equiv \overline{\gamma}+\overline{m}\,\overline{\alpha}_1/r$.

Next find conditions on $(\underline{\alpha}_1,\overline{\alpha}_1,c_a,c_b)$ such that $\underline{\alpha}(\cdot)$ and $\overline{\alpha}(\cdot)$ are lower and upper solutions to Eq. (29) when $X\in(-1,1)$. From Eq. (32), $\underline{\alpha}''(X)=-\frac{3}{2}\underline{\alpha}_1(1-X^2)\geq -\frac{3}{2}\underline{\alpha}_1$ and $\underline{\alpha}(X)\leq -\frac{3}{8}\underline{\alpha}_1-c_a$, and from Eq. (33) $\overline{\alpha}''(X)=\frac{3}{2}\overline{\alpha}_1(1-X^2)\leq\frac{3}{2}\overline{\alpha}_1$ and $\overline{\alpha}(X)\geq\frac{3}{8}\overline{\alpha}_1+c_b$ when $X\in(-1,1)$. Substituting this into Eqs. (34) and (35), this corresponds to finding $\underline{\alpha}_1,\overline{\alpha}_1,c_a,c_b\geq 0$ such that

$$c_a\geq\frac{3}{4}\left(\frac{|\sigma^*(X,\underline{\alpha}'(X))|^2}{r}-\frac{1}{2}\right)\underline{\alpha}_1-\psi(X,\underline{\alpha}'(X)) \tag{36}$$

$$c_b\geq\frac{3}{4}\left(\frac{|\sigma^*(X,\overline{\alpha}'(X))|^2}{r}-\frac{1}{2}\right)\overline{\alpha}_1+\psi(X,\overline{\alpha}'(X)). \tag{37}$$

Let $\overline{\sigma}\equiv\sup_{X\in[0,1],z\in\mathbb{R}}\sigma^*(X,z)$, which exists since $\sigma^*(X,z)$ is Lipschitz continuous in $X$ and bounded in $z$ (the latter follows from $B(X)$ bounded on $[0,1]$, which implies $b^*(X,z)$ is bounded on $[0,1]\times\mathbb{R}$, and $\sigma(b,X)$ Lipschitz continuous). First consider $X\in(-1,0]$, which means that $\overline{\alpha}'(X)=\frac{1}{2}\overline{\alpha}_1X\,(3-X^2)\in(-\overline{\alpha}_1,0]$ and $\underline{\alpha}'(X)=-\frac{1}{2}\underline{\alpha}_1X\,(3-X^2)\in[0,\underline{\alpha}_1)$. From the bound on $\psi(X,z)$,

$$\begin{aligned}\psi(X,\underline{\alpha}'(X)) &\geq \left(\underline{g}_1+\frac{\underline{\mu}_1}{r}\underline{\alpha}'\right)X+\underline{\gamma}+\frac{\underline{m}}{r}\underline{\alpha}'\geq -\underline{g}_1+\underline{\gamma}-\frac{\underline{\mu}_1}{r}\underline{\alpha}_1+\frac{\underline{\alpha}_1}{r}\min\{\underline{m},0\}\\ \psi(X,\overline{\alpha}'(X)) &\leq \left(-\overline{g}_1+\frac{\underline{\mu}_1}{r}\overline{\alpha}'\right)X+\overline{\gamma}+\frac{\underline{m}}{r}\overline{\alpha}'\leq \overline{g}_1+\overline{\gamma}+\frac{\underline{\mu}_1}{r}\overline{\alpha}_1-\frac{\overline{\alpha}_1}{r}\min\{\underline{m},0\}.\end{aligned}$$

Therefore, when $X\in(-1,0]$, Eqs. (36) and (37) hold when

$$\begin{aligned}c_a &\geq c_{a3}\equiv\frac{3}{4}\left(\frac{\overline{\sigma}^2}{r}-\frac{1}{2}\right)\underline{\alpha}_1+\underline{g}_1-\underline{\gamma}+\frac{\underline{\mu}_1}{r}\underline{\alpha}_1-\frac{\underline{\alpha}_1}{r}\min\{\underline{m},0\}\\ c_b &\geq c_{b3}\equiv\frac{3}{4}\left(\frac{\overline{\sigma}^2}{r}-\frac{1}{2}\right)\overline{\alpha}_1+\overline{g}_1+\overline{\gamma}+\frac{\underline{\mu}_1}{r}\overline{\alpha}_1-\frac{\overline{\alpha}_1}{r}\min\{\underline{m},0\}.\end{aligned}$$

Next consider $X\in[0,1)$, which means that $\overline{\alpha}'(X)=\frac{1}{2}\overline{\alpha}_1X\,(3-X^2)\in[0,\overline{\alpha}_1)$ and $\underline{\alpha}'(X)=$

$-\frac{1}{2}\underline{\alpha}_1 X\left(3-X^2\right) \in (-\underline{\alpha}_1, 0]$. From the bound on $\psi(X,z)$,

$$\begin{aligned}
\psi(X,\underline{\alpha}'(X)) &\geq \left(-\underline{g}_2 + \frac{\overline{\mu}_2}{r}\underline{\alpha}'\right) X + \underline{\gamma} + \frac{\overline{m}}{r}\underline{\alpha}' \geq -\underline{g}_2 + \underline{\gamma} - \frac{\overline{\mu}_2}{r}\underline{\alpha}_1 - \frac{\underline{\alpha}_1}{r}\max\left\{\overline{m},0\right\} \\
\psi(X,\overline{\alpha}'(X)) &\leq \left(\overline{g}_2 + \frac{\overline{\mu}_2}{r}\overline{\alpha}'\right) X + \overline{\gamma} + \frac{\overline{m}}{r}\overline{\alpha}' \leq \overline{g}_2 + \overline{\gamma} + \frac{\overline{\mu}_2}{r}\overline{\alpha}_1 + \frac{\overline{\alpha}_1}{r}\max\left\{\overline{m},0\right\}.
\end{aligned}$$

Therefore, when $X \in [0,1)$, Eqs. (36) and (37) hold when

$$\begin{aligned}
c_a &\geq c_{a4} \equiv \frac{3}{4}\left(\frac{\overline{\sigma}^2}{r} - \frac{1}{2}\right)\underline{\alpha}_1 + \underline{g}_2 - \underline{\gamma} + \frac{\overline{\mu}_2}{r}\underline{\alpha}_1 + \frac{\underline{\alpha}_1}{r}\max\left\{\overline{m},0\right\} \\
c_b &\geq c_{b4} \equiv \frac{3}{4}\left(\frac{\overline{\sigma}^2}{r} - \frac{1}{2}\right)\overline{\alpha}_1 + \overline{g}_2 + \overline{\gamma} + \frac{\overline{\mu}_2}{r}\overline{\alpha}_1 + \frac{\overline{\alpha}_1}{r}\max\left\{\overline{m},0\right\}.
\end{aligned}$$

Combine these conditions, choosing

$$\begin{aligned}
\underline{\alpha}_1 &\equiv \max\left\{\frac{r\underline{g}_1}{r-\underline{\mu}_1}, \frac{r\underline{g}_2}{r-\overline{\mu}_2}\right\} \\
\overline{\alpha}_1 &\equiv \max\left\{\frac{r\overline{g}_1}{r-\underline{\mu}_1}, \frac{r\overline{g}_2}{r-\overline{\mu}_2}\right\}
\end{aligned}$$

yields $\underline{\alpha}_1 \geq 0$ and $\overline{\alpha}_1 \geq 0$ that satisfy all of the slope conditions. Choosing $c_a \equiv \max\left\{0, c_{a1}, c_{a2}, c_{a3}, c_{a4}\right\}$ and $c_b \equiv \max\left\{0, c_{b1}, c_{b2}, c_{b3}, c_{b4}\right\}$ yields $c_a \geq 0$ and $c_b \geq 0$ that satisfy all of the intercept conditions. Given $\underline{\alpha}_1, \overline{\alpha}_1, c_a, c_b$, the functions $\underline{\alpha}(\cdot)$ and $\overline{\alpha}(\cdot)$ defined in Eqs. (32) and (33) are lower and upper solutions to Eq. (29).

*Step 1b.* Show that there exist $\underline{\alpha}_1, \overline{\alpha}_1, c_a, c_b \geq 0$ such that $\underline{\alpha}(\cdot)$ and $\overline{\alpha}(\cdot)$ are lower and upper solutions to Eq. (29) when $g$ is bounded. Define $\overline{g} \equiv \sup_{(a,b,X)\in A\times E} g(a,b,X)$ and $\underline{g} \equiv \inf_{(a,b,X)\in A\times E} g(a,b,X)$, which exist since $g$ is bounded. Let $\underline{\alpha}_1 = 0$ and $c_a = -\underline{g}$. Then $\psi(X,\underline{\alpha}'(X)) = g^*(X,0)$, so $\underline{\alpha}(X) - \psi(X,\underline{\alpha}'(X)) = \underline{g} - g^*(X,0) \leq 0$ for all $X$ and $\underline{\alpha}\left(X\right) = \underline{g}$ is a lower solution. Similarly, let $\overline{\alpha}_1 = 0$ and $c_b = \overline{g}$. Then $\psi(X,\overline{\alpha}'(X)) = g^*(X,0)$, so $\overline{\alpha}(X) - \psi(X,\overline{\alpha}'(X)) = \overline{g} - g^*(X,0) \geq 0$ for all $X$ and $\overline{\alpha}(X) = \overline{g}$ is an upper solution. Note that this step places no restrictions on the growth rate of $\mu$ in relation to $r$.

*Step 2.* Show that the Nagumo condition Eq. (31) is satisfied. Given a bounded interval $I \subset \mathcal{X}$, there exists a $K_I > 0$ such that

$$|f(X,U,U')| = \left|\frac{2r}{\sigma^*(X,U')^2}\left(U - g^*(X,U') - \frac{U'}{r}\mu^*(X,U')\right)\right| \leq K_I(1+|U'|)$$

for all $(X,U,U') \in \{I \times \mathbb{R}^2 \text{ s.t. } \underline{\alpha}(X) \leq U \leq \overline{\alpha}(X)\}$. This follows directly from the fact that

$X \in I$, $\underline{\alpha}(X)$ and $\overline{\alpha}(X)$ are bounded on $I$, $\underline{\alpha}(X) \leq U \leq \overline{\alpha}(X)$, $g^*$ and $\mu^*$ are bounded on $(X, U') \in I \times \mathbb{R}$ and $\sigma(b, X)$ is bounded away from zero on $I$. Define $H_I(z) = K_I(1 + z)$. Therefore, $\int_0^\infty z/H_I(z)dz = \infty$.

Conclude that Eq. (29) has at least one $\mathcal{C}^2$ solution $U$ such that for all $X \in \mathbb{R}$, $\underline{\alpha}(X) \leq U(X) \leq \overline{\alpha}(X)$. In the case where $g$ is unbounded, the $\underline{\alpha}(X)$ and $\overline{\alpha}(X)$ constructed in Step 1a have linear growth, so $U$ has linear growth. In the case where $g$ is bounded, the $\underline{\alpha}(X)$ and $\overline{\alpha}(X)$ constructed in Step 1a are bounded, so $U$ is bounded. □

The proof of Lemma 3 relies on a result from Faingold and Sannikov (2011), which is reproduced below in a slightly altered form to apply to the current setting.

**Lemma 20** (Faingold Sannikov (2011))**.** *Let $D = \{(t, u, v) \in (\underline{t}, \overline{t}) \times \mathbb{R}^2\}$ and $f : D \to \mathbb{R}$ be continuous. Let $\underline{\alpha}_1, \overline{\alpha}_1 \in \mathbb{R}$ be constants such that $\underline{\alpha}_1 \leq \overline{\alpha}_1$ and $f(t, \underline{\alpha}_1, 0) \leq 0 \leq f(t, \overline{\alpha}_1, 0)$ for all $t \in \mathbb{R}$. Assume that for any closed interval $I \subset (\underline{t}, \overline{t})$, there exists a $K_I > 0$ such that $|f(t, u, v)| \leq K_I(1 + |v|))$ for all $(t, u, v) \in I \times [\underline{\alpha}_1, \overline{\alpha}_1] \times \mathbb{R}$. Then the differential equation $U'' = f(t, U(t), U'(t))$ has at least one $\mathcal{C}^2$ solution $U$ on $(\underline{t}, \overline{t})$ such that $\underline{\alpha}_1 \leq U(t) \leq \overline{\alpha}_1$.*

**Proof of Lemma 3.** Suppose $\mathcal{X}$ is compact. Then Eq. (30) is continuous on the set $D = \left\{(X, U, U') \in (\underline{X}, \overline{X}) \times \mathbb{R}^2\right\}$. When $\mathcal{X}$ is compact, the feasible payoff set for the large player is bounded, since $g$ is Lipschitz continuous. Define $\underline{g} \equiv \inf_{(a,b,X) \in A \times E} g(a, b, X)$ and $\overline{g} \equiv \sup_{(a,b,X) \in A \times E} g(a, b, X)$ as the lower and upper bound on the flow payoff for the large player, respectively. For any closed interval $I \subset (\underline{X}, \overline{X})$, there exists a $K_I > 0$ such that

$$\left| \frac{2r}{\sigma^*(X, U')^2} \left( U - g^*(X, U') - \frac{U'}{r} \mu^*(X, U') \right) \right| \leq K_I(1 + |U'|)$$

for all $(X, U, U') \in I \times [\underline{g}, \overline{g}] \times \mathbb{R}$. This follows directly from the fact that $X \in I$, $U \in [\underline{g}, \overline{g}]$, $g^*$ and $\mu^*$ are bounded on $\mathcal{X} \times \mathbb{R}$, and $\sigma(b, X)$ is bounded away from zero on $I$. Also note that

$$f(X, \underline{g}, 0) = \frac{2r}{\sigma^*(X, 0)^2} (\underline{g} - g^*(X, 0)) \leq 0 \leq f(X, \overline{g}, 0) = \frac{2r}{\sigma^*(X, 0)^2} (\overline{g} - g^*(X, 0))$$

for all $X \in (\underline{X}, \overline{X})$. By Lemma 20, Eq. (29) has at least one $\mathcal{C}^2$ bounded solution $U$ on $(\underline{X}, \overline{X})$ with $\underline{g} \leq U(X) \leq \overline{g}$. This establishes that there exists a bounded solution to the optimality equation Eq. (7). □

# C Additional Material for Theorem 2

This section establishes Lemma 5 for the case of $\mathcal{X} = [\underline{X}, \overline{X}]$. Let $U$ be a bounded solution to Eq. (7). Since $U$ is not defined at $X \in \{\underline{X}, \overline{X}\}$, take the definitions of $d(X, \beta)$ and

$f(X,\beta)$ from Appendix A.3 for $X \in (\underline{X}, \overline{X})$ and define these functions at the boundary as follows: $d(\underline{X},\beta) = d(\overline{X},\beta) = 0$ and $f\underline{X},\beta) = f(\overline{X},\beta) = 0$. Note that $\sigma(X, r\beta)$ is bounded away from 0 on any compact proper subset $I \subset [\underline{X}, \overline{X}]$ by Assumption 1, so by Lemma 4, if $f(X,\beta) = 0$ for some $X \in I$, then $d(X,\beta) = 0$. The following establishes that for every $\varepsilon > 0$, there exists a $\eta > 0$ such that either $d(X,\beta) > -\varepsilon$ or $|f(X,\beta)| > \eta$. Fix any $\varepsilon > 0$. First show that there exists a $\delta > 0$ such that $|d(X,\beta)| < \varepsilon$ for $(X,\beta) \in \Omega_a \equiv \{\mathcal{X} \times \mathbb{R} : |X \notin [\underline{X}+\delta, \overline{X}-\delta]\}$, which establishes the claim. Given $\mu(a,b,\overline{X}) = \overline{m}$ for all $(a,b) \in A \times B(\overline{X})$, from Eq. (6), $S^*(\overline{X},z) = S^*(\overline{X},0)$ for all $z \in \mathbb{R}$. Therefore, $g^*(\overline{X},z) = g^*(\overline{X},0)$. Moreover, $\mu^*(\overline{X},z) = \overline{m}$ and $\sigma^*(\overline{X},z) = 0$ for all $z \in \mathbb{R}$ by assumption. Given Lipschitz constant $K_g > 0$ for $g^*$, $|g^*(X,z) - g^*(\overline{X},0)| = |g^*(X,z) - g^*(\overline{X},z)| \leq K_g|\overline{X} - X|$ for all $z \in \mathbb{R}$. Choosing $\delta_1 = \varepsilon/8rK_g$, if $|\overline{X} - X| < \delta_1$ then $|g^*(X,z) - g^*(\overline{X},0)| < \varepsilon/8r$ for all $z \in \mathbb{R}$. This implies $r|g^*(X,U'(X)) - g^*(X,r\beta)| < \varepsilon/4$ for all $|\overline{X} - X| < \delta_1$ and $\beta \in \mathbb{R}$. When $\overline{m} \neq 0$, $U'(X)$ is bounded by Lemma 23. Let $M > 0$ denote this bound. Analogously, there exists a $\delta_2 > 0$ such that $|\mu^*(X,U'(X)) - \mu^*(X,r\beta)| < \varepsilon/4M$ for all $|\overline{X} - X| < \delta_2$ and $\beta \in \mathbb{R}$. Therefore, $|\mu^*(X,U'(X)) - \mu^*(X,r\beta)||U'(X)| < \varepsilon/4$ for all $|\overline{X} - X| < \delta_2$ and $\beta \in \mathbb{R}$. When $\overline{m} = 0$, then $(\overline{X} - X)U'(X) \to 0$ by Lemma 23. Note Lipschitz continuity and $\mu^*(\overline{X},z) = 0$ implies $|\mu^*(X,z)| \leq K_\mu(\overline{X} - X)$ for all $z \in \mathbb{R}$. Therefore, there exists a $\delta_3 > 0$ such that $(\overline{X} - X)U'(X) < \varepsilon/8K_\mu$ for all $|\overline{X} - X| < \delta_3$. Then $|\mu^*(X,z)||U'(X)| \leq K_\mu(\overline{X} - X)|U'(X)| < \varepsilon/8$ for all $|\overline{X} - X| < \delta_3$ and $z \in \mathbb{R}$. This implies $|\mu^*(X,U'(X)) - \mu^*(X,r\beta)||U'(X)| < \varepsilon/4$ for all $|\overline{X} - X| < \delta_3$ and $\beta \in \mathbb{R}$. Finally, by Assumption 1, there exists a $K_{1,\sigma} > 0$ such that $\sigma^*(X,z) \geq K_{1,\sigma}(\overline{X} - X)$ for all $z \in \mathbb{R}$ and $X < (\overline{X} - \underline{X})/2$. Therefore, by Lemma 24, $K^2_{1,\sigma}(\overline{X}-X)^2|U''(X)| \leq \sigma^*(X,U'(X))^2|U''(X)| \to 0$ as $X \to \overline{X}$. By Lipschitz continuity and $\sigma^*(\overline{X},z) = 0$, there exists a $K_{2,\sigma} > 0$ such that $\sigma^*(X,z) \leq K_{2,\sigma}(\overline{X} - X)$ for all $z \in \mathbb{R}$. Taken together, this implies that there exists a $\delta_4 > 0$ such that $\sigma^*(X,r\beta)^2|U''(X)| \leq K^2_{2,\sigma}(\overline{X} - X)^2|U''(X)| < \varepsilon/4$ for $|\overline{X} - X| < \delta_4$. Therefore, $|\sigma^*(X,U'(X))^2 - \sigma^*(X,r\beta)^2||U''(X)|/2 < \varepsilon/4$ for $|\overline{X} - X| < \delta_4$. Taken together, this implies that $|d(X,\beta)| < 3\varepsilon/4$ for $|\overline{X} - X| < \delta_H$, where $\delta_H \equiv \min\{\delta_1, \delta_2, \delta_4\}$ when $\overline{m} \neq 0$ and $\delta_H \equiv \min\{\delta_1, \delta_3, \delta_4\}$ when $\overline{m} = 0$. Analogously, there exists a $\delta_L > 0$ such that $|d(X,\beta)| < 3\varepsilon/4$ for $|\underline{X} - X| < \delta_L$. Taking $\delta \equiv \min\{\delta_L, \delta_H\}$ establishes $d(X,\beta) > -\varepsilon$ for all $(X,\beta) \in \Omega_a$. Next show that there exists an $M > 0$ such that this is true for $(X,\beta) \in \Omega_b \equiv \{\mathcal{X} \times \mathbb{R} : |\beta| > M, X \in [\underline{X}+\delta, \overline{X}-\delta]\}$. On any compact proper subset of $[\underline{X}, \overline{X}]$, $U'$ is bounded and $\sigma(\bar{b}, X)$ is bounded away from 0. Therefore, there exists an $M > 0$ and $\eta_1 > 0$ such that $|f(X,\beta)| > \eta_1$ for all $|\beta| > M$ and $X \in [\underline{X}+\delta, \overline{X}-\delta]$. Finally show this is true for $(X,\beta) \in \Omega_c \equiv \{\mathcal{X} \times \mathbb{R} : |\beta| \leq M, X \in [\underline{X}+\delta, \overline{X}-\delta]\}$. Consider the set $\Phi_c \subset \Omega_c$ where $d(X,\beta) \leq -\varepsilon$. The function $d$ is continuous and $\Omega_c$ is compact, so $\Phi_c$ is compact. The function $|f|$ is continuous, and therefore achieves a minimum $\eta_3$ on $\Phi_c$. If

$\eta_3 = 0$, then $d = 0$ by Lemma 4 (since $\sigma(\overline{b}, X)$ is bounded away from 0 on $[\underline{X} + \delta, \overline{X} - \delta]$), a contradiction. Therefore, $\eta_3 > 0$. Take $\eta \equiv \min\{\eta_1, \eta_2, \eta_3\}$. Then when $d(X, \beta) \leq -\varepsilon$, $|f(X, \beta)| > \eta$.

# D Additional Material for Theorems 3 and 4

This section contains additional material used to establish the boundary conditions in the proofs of Theorems 3 and 4.

## D.1 Boundary conditions for compact $\mathcal{X}$.

When $\mathcal{X}$ is compact, Lemmas 21 to 25 establish the following boundary conditions under Assumptions 1 to 3. When $\overline{X}$ is an absorbing state ($\overline{m} = 0$), any bounded solution $U$ of Eq. (7) on $(\underline{X}, \overline{X})$ satisfies $\lim_{X \to p} U(X) = g^*(p, 0)$, $\lim_{X \to p} \mu^*(X, U'(X))U'(X) = 0$, and $\lim_{X \to p} \sigma^*(X, U'(X))^2 U''(X) = 0$ for $p \in \{\underline{X}, \overline{X}\}$. When $\overline{X}$ is not an absorbing state ($\overline{m} \neq 0$), any bounded solution $U$ of Eq. (7) on $(\underline{X}, \overline{X})$ satisfies $\lim_{X \to \overline{X}} U(X) = g^*(\overline{X}, 0) + \overline{m}\ \overline{u}'/r$, $\lim_{X \to \overline{X}} \mu^*(X, U'(X))U'(X) = \overline{m}\ \overline{u}'$, and $\lim_{X \to \overline{X}} \sigma^*(X, U'(X))^2 U''(X) = 0$ given finite $\overline{u}' \equiv \lim_{X \to \overline{X}} U'(X)$, with analogous conditions for $X \to \underline{X}$.

**Lemma 21.** *Suppose $\mathcal{X}$ is compact. Any bounded solution $U$ of Eq. (7) on $(\underline{X}, \overline{X})$ has bounded variation and* $\liminf_{X \to p} U'(X) = \limsup_{X \to p} U'(X)$ *for* $p \in \{\underline{X}, \overline{X}\}$.

*Proof.* Suppose $U$ is a bounded solution of Eq. (7) with unbounded variation near $p = \overline{X}$. Then there exists an increasing sequence $(X_n)_{n \in \mathbb{N}}$ of alternating consecutive local maxima and minima of $U$, with $U'(X_n) = 0$ and $U''(X_n) \leq 0$ for the maxima and $U'(X_n) = 0$ and $U''(X_n) \geq 0$ for the minima. Given Eq. (6), a static Nash equilibrium is played at any $X$ such that $U'(X) = 0$, yielding flow payoff $g^*(X, 0)$. From Eq. (7), this implies $g^*(X_n, 0) \geq U(X_n)$ in the case of a maximum and $g^*(X_n, 0) \leq U(X_n)$ in the case of a minimum. Thus, the total variation of $g^*(X, 0)$ on $[X_1, \overline{X})$ is at least as large as the total variation of $U$, and therefore, $g^*(X, 0)$ has unbounded variation near $\overline{X}$. This is a contradiction since $g^*(\cdot, 0)$ is Lipschitz continuous by Assumption 3.

Next show $\liminf_{X \to \overline{X}} U'(X) = \limsup_{X \to \overline{X}} U'(X)$. Suppose not. Then by the continuity of $U'$, there exists a $z$ and an increasing sequence $(X_n)_{n \in \mathbb{N}}$ of alternating consecutive $X$ with $U'(X_n) = z$ and $U''(X_n) \leq 0$ for $n$ odd and $U'(X_n) = z$ and $U''(X_n) \geq 0$ for $n$ even, with one inequality for $U''$ strict. From Eq. (7), this implies $U(X_n) \leq \psi(X_n, z)$ for $n$ odd and $\psi(X_n, z) \leq U(X_n)$ for $n$ even, with one inequality strict. Thus, the total variation of $\psi(X, z)$ on $[X_1, \overline{X})$ is at least as large as the total variation of $U$, and therefore, $\psi(X, z)$ has unbounded variation near $\overline{X}$. This is a contradiction since $g^*(\cdot, z)$ and $\mu^*(\cdot, z)$ are Lipschitz continuous by Assumption 3, and therefore, for any fixed $z$, $\psi(\cdot, z)$ is Lipschitz continuous.

Therefore, it must be that $\liminf_{X\to\overline{X}} U'(X) = \limsup_{X\to\overline{X}} U'(X)$. Let $\overline{u}' \equiv \lim_{X\to\overline{X}} U'(X)$ denote this (possibly infinite) limit. The case of $p = \underline{X}$ is analogous. $\square$

**Lemma 22.** *Suppose $\mathcal{X}$ is compact and $f : \mathcal{X} \to \mathbb{R}$ is Lipschitz continuous with $f(\overline{X}) = f(\underline{X}) = 0$. Then for $p \in \{\underline{X}, \overline{X}\}$, any bounded solution $U$ of Eq. (7) on $(\underline{X}, \overline{X})$ satisfies*

$$\begin{aligned}
\liminf_{X\to p} |f(X)|U'(X) &\leq 0 \leq \limsup_{X\to p} |f(X)|U'(X) \\
\liminf_{X\to p} f(X)^2 U''(X) &\leq 0 \leq \limsup_{X\to p} f(X)^2 U''(X).
\end{aligned}$$

*Proof.* Consider $p = \overline{X}$ and suppose $\liminf_{X\to\overline{X}} |f(X)|U'(X) > 0$. Then there exists a $\delta > 0$ and $\varepsilon > 0$ such that for all $X \in (\overline{X} - \delta, \overline{X})$, $|f(X)|U'(X) > \varepsilon$. By Lipschitz continuity, there exists an $M > 0$ such that $|f(X)| \leq M(\overline{X} - X)$ for $X \in \mathcal{X}$. Together this implies $U'(X) > \varepsilon/|f(X)| \geq \varepsilon/(M(\overline{X} - X))$ for all $X \in (\overline{X} - \delta, \overline{X})$. The antiderivative of $\varepsilon/(M(\overline{X} - X))$ is $-(\varepsilon/M)\ln(\overline{X} - X)$, which converges to $\infty$ as $X \to \overline{X}$. This contradicts the boundedness of $U$. Therefore it must be that $\liminf_{X\to\overline{X}} |f(X)|U'(X) \leq 0$. The proof is analogous to show $\limsup_{X\to\overline{X}} |f(X)|U'(X) \geq 0$ and for the case of $p = \underline{X}$.

Suppose $\liminf_{X\to\overline{X}} f(X)^2 U''(X) > 0$. There exists a $\delta_2 > 0$ and $\varepsilon_2 > 0$ such that for all $X \in (\overline{X} - \delta_2, \overline{X})$, $f(X)^2 U''(X) > \varepsilon_2$. Then for all $X \in (\overline{X} - \delta_2, \overline{X})$, $U''(X) > \varepsilon_2/f(X)^2 > \varepsilon_2/M^2(\overline{X} - X)^2$. The second-order antiderivative of $\varepsilon_2/M^2(\overline{X} - X)^2$ is $-(\varepsilon_2/M^2)\ln(\overline{X} - X)$ which converges to $\infty$ as $X \to \overline{X}$. This contradicts the boundedness of $U$. Therefore $\liminf_{X\to\overline{X}} f(X)^2 U''(X) \leq 0$. The proof is analogous to show $\limsup_{X\to\overline{X}} f(X)^2 U''(X) \geq 0$ and for the case of $p = \underline{X}$. $\square$

**Lemma 23.** *Suppose $\mathcal{X}$ is compact. Any bounded solution $U$ of Eq. (7) on $(\underline{X}, \overline{X})$ satisfies $\lim_{X\to\overline{X}} \mu^*(X, U'(X))U'(X) = 0$ when $\overline{X}$ is an absorbing state ($\overline{m} = 0$) and $\lim_{X\to\overline{X}} \mu^*(X, U'(X))U'(X) = \overline{m}\,\overline{u}'$ for some finite $\overline{u}' \equiv \lim_{X\to\overline{X}} U'(X)$ when $\overline{X}$ is not an absorbing state ($\overline{m} \neq 0$), with analogous limits as $X \to \underline{X}$.*

*Proof.* Consider $p = \overline{X}$. By Lemma 21, $\liminf_{X\to\overline{X}} U'(X) = \limsup_{X\to\overline{X}} U'(X)$. Let $\overline{u}' \equiv \lim_{X\to\overline{X}} U'(X)$. First show that when $\overline{m} \neq 0$, $|\overline{u}'| < \infty$. Suppose not. Note that $\lim_{X\to\overline{X}} \mu^*(X, U'(X)) = \overline{m}$ follows from the Lipschitz continuity of $\mu^*$ and $\mu^*(\overline{X}, z) = \overline{m}$ for all $z$. Then if $|\overline{u}'| = \infty$ and $\overline{m} \neq 0$, $\lim_{X\to\overline{X}} |\mu^*(X, U'(X))U'(X)| = \infty$. From Eq. (7), this implies

$$|\sigma^*(X, U'(X))^2 U''(X)| = |2r(U(X) - g^*(X, U'(X))) - 2\mu^*(X, U'(X))U'(X)| \to \infty$$

since $U$ and $g^*$ are bounded. But given that $\sigma^*$ is Lipschitz continuous with $\sigma^*(\overline{X}, z) = 0$ for all $z$, this contradicts Lemma 22. Therefore, it must be that $|\overline{u}'| < \infty$ when $\overline{m} \neq 0$. Taken together, this implies $\lim_{X\to\overline{X}} \mu^*(X, U'(X))U'(X) = \overline{m}\,\overline{u}'$ when $\overline{m} \neq 0$.

Next show that when $\overline{m} = 0$, $\lim_{X\to\overline{X}} \mu^*(X, U'(X))U'(X) = 0$. Let $f(X) \equiv \overline{X} - X$ and first show $\lim_{X\to\overline{X}} f(X)U'(X) = 0$. Suppose $\limsup_{X\to\overline{X}} f(X)U'(X) > 0$. By Lemma 22, $\liminf_{X\to\overline{X}} f(X)U'(X) \leq 0$ since $f(X)$ is Lipschitz continuous and $f(\overline{X}) = 0$. Then there exist constants $K > k > 0$ such that $f(X)U'(X)$ crosses $k$ and $K$ infinitely many times in a neighborhood of $\overline{X}$. Additionally, there exist $\varepsilon > 0$ and $L > 0$ such that for $X$ with $\overline{X} - X < \varepsilon$ and $f(X)U'(X) \in (k, K)$,

$$|U''(X)| = \left| \frac{2r(U(X) - g^*(X, U'(X))) - 2\mu^*(X, U'(X))U'(X)}{\sigma^*(X, U'(X))^2} \right| \leq \frac{L}{f(X)^2}.$$

where the equality follows from Eq. (7) and the inequality follows from the triangle inequality, $U$ and $g^*$ bounded, $\mu^*(X, U'(X)) \leq K_\mu f(X)$ for some $K_\mu > 0$ by Lipschitz continuity, and $\sigma^*(X, U'(X)) \geq K_\sigma f(X)$ for some $K_\sigma > 0$ when $X < (\overline{X} - \underline{X})/2$ by Assumption 1. Given $f$ is Lipschitz continuous, there exists an $L_2 > 0$ such that $|f'(X)| < L_2$. This implies that for $X$ such that $|\overline{X} - X| < \varepsilon$ and $|f(X)|U'(X) \in (k, K)$,

$$\begin{aligned} |(f(X)U'(X))'| &\leq |f'(X)U'(X)| + |f(X)U''(X)| = \left( |f'(X)| + \frac{|f(X)^2 U''(X)|}{|f(X)U'(X)|} \right) |U'(X)| \\ &\leq (L_2 + L/k)|U'(X)|, \end{aligned}$$

and therefore, $|U'(X)| \geq |(f(X)U'(X))'|/(L_2 + L/k)$. Therefore, the total variation of $U$ is at least $(K-k)/(L_2+L/k) > 0$ on any interval where $|f(X)|U'(X)$ crosses $k$ and stays in $(k, K)$ until crossing $K$. This happens infinitely often in a neighborhood of $\overline{X}$, which implies that $U$ has unbounded variation in this neighborhood. This is a contradiction by Lemma 21. Thus, $\limsup_{X\to\overline{X}} |f(X)|U'(X) = 0$. By similar logic, $\liminf_{X\to\overline{X}} |f(X)|U'(X) = 0$, and therefore, $\lim_{X\to\overline{X}} |f(X)|U'(X) = 0$.[3] Given $|\mu^*(X, U'(X))| \leq K_1 f(X)$, this implies $|\mu^*(X, U'(X))U'(X)| \leq K_1|f(X)U'(X)| \to 0$ as $X \to \overline{X}$. The case of $p = \underline{X}$ is analogous. $\square$

**Lemma 24.** *Suppose $\mathcal{X}$ is compact. Any bounded solution $U$ of Eq. (7) on $(\underline{X}, \overline{X})$ satisfies $\lim_{X\to p} \sigma^*(X, U'(X))^2 U''(X) = 0$ for $p \in \{\underline{X}, \overline{X}\}$. When $\overline{X}$ is an absorbing state ($\overline{m} = 0$), $\lim_{X\to\overline{X}} U(X) = g^*(\overline{X}, 0)$ and when $\overline{X}$ is not an absorbing state ($\overline{m} \neq 0$), $\lim_{X\to\overline{X}} U(X) = g^*(\overline{X}, 0) + \overline{m}\,\overline{u}'$ given finite $\overline{u}' \equiv \lim_{X\to\overline{X}} U'(X)$, with analogous limits as $X \to \underline{X}$.*

*Proof.* Consider $p = \overline{X}$. Given that $U$ is continuous, bounded, and has bounded variation,

[3] This result holds under a more general condition than $\sigma^*(X, z) \geq C(\overline{X} - X)(X - \underline{X})$ for all $(X, z) \in \mathcal{X} \times \mathbb{R}$. Specifically, for any positive Lipschitz continuous function $f(X)$ with $f(\overline{X}) = 0$ and a $K_1, K_2, \delta > 0$ such that $|\mu^*(X, z)| \leq K_1 f(X)$ and $\sigma^*(X, z) \geq K_2 f(X)$ for $X$ such that $|\overline{X} - X| < \delta$ and $z \in \mathbb{R}$, then $\lim_{X\to\overline{X}} f(X)U'(X) = 0$ for all $z \in \mathbb{R}$. This condition relates the growth rate of $\sigma^*$ to that of $\mu^*$. When $\sigma^*(X, z) \geq C(\overline{X} - X)(X - \underline{X})$, the Lipschitz continuity of $\mu^*$ implies $f(X) = \overline{X} - X$ satisfies this condition.

$U_{\overline{X}} \equiv \lim_{X\to\overline{X}} U(X)$ exists. Given $\mu(a,b,\overline{X}) = \overline{m}$ for all $(a,b) \in A \times B(\overline{X})$, from Eq. (6), $S^*(\overline{X},z) = S^*(\overline{X},0)$ for all $z \in \mathbb{R}$. Therefore, $g^*(\overline{X},z) = g^*(\overline{X},0)$ for all $z \in \mathbb{R}$. By the Lipschitz continuity of $g^*$, this implies $\lim_{X\to\overline{X}} g^*(X,U'(X)) = g^*(\overline{X},0)$.

First suppose $\overline{X}$ is an absorbing state ($\overline{m} = 0$). By Lemma 23, $\lim_{X\to\overline{X}} \mu^*(X,U'(X))U'(X) = 0$. Plugging these limits into Eq. (7),

$$\begin{aligned}\lim_{X\to\overline{X}} \sigma^*(X,U'(X))^2U''(X) &= \lim_{X\to\overline{X}} 2r(U(X) - g^*(X,U'(X))) - 2\mu^*(X,U'(X))U'(X)) \\ &= 2r(U_{\overline{X}} - g^*(\overline{X},0)).\end{aligned}$$

Suppose $U_{\overline{X}} > g^*(\overline{X},0)$. By $\mathcal{X}$ compact, $\sigma(b,\overline{X}) = 0$ for all $b \in B(\overline{X})$, and therefore, $\sigma^*(\overline{X},z) = 0$ for all $z \in \mathbb{R}$. Therefore, by the Lipschitz continuity of $\sigma^*$, there exists an $M > 0$ such that for all $X \in \mathcal{X}$, $\sigma^*(X,U'(X))^2 \leq M(\overline{X} - X)^2$. This implies

$$\lim \inf_{X\to\overline{X}} \sigma^*(X,U'(X))^2|U''(X)| = 2r(U_{\overline{X}} - g^*(\overline{X},0)) \leq \lim \inf_{X\to\overline{X}} M(\overline{X} - X)^2|U''(X)| = 0$$

where the last equality follows from Lemma 22. This is a contradiction. A similar contradiction holds for $U_{\overline{X}} < g^*(\overline{X},0)$. Therefore, $U_{\overline{X}} = g^*(\overline{X},0)$. Then $\lim_{X\to\overline{X}} \sigma^*(X,U'(X))^2U''(X) = 0$ follows immediately from Eq. (7).

Next suppose $\overline{X}$ is not an absorbing state ($\overline{m} \neq 0$). By similar reasoning, $\lim_{X\to\overline{X}} \sigma^*(X,U'(X))^2U''(X) = 2r(U_{\overline{X}} - g^*(\overline{X},0) - \overline{m}\,\overline{u}'/r)$, which yields $U_{\overline{X}} = g^*(\overline{X},0) + \overline{m}\,\overline{u}'/r$ and again, $\lim_{X\to\overline{X}} \sigma^*(X,U'(X))^2U''(X) = 0$. The case of $p = \underline{X}$ is analogous. □

**Lemma 25.** *Suppose $U$ and $V$ are bounded solutions of Eq. (7) on $(\underline{X},\overline{X})$. Then $\lim_{X\to p} V(X) - U(X) = 0$ for $p \in \{\underline{X},\overline{X}\}$.*

*Proof.* Consider $p = \overline{X}$. Let $U$ and $V$ be bounded solutions of Eq. (7). When $\overline{X}$ is an absorbing state ($\overline{m} = 0$), $\lim_{X\to\overline{X}} V(X) - U(X) = 0$ follows immediately from Lemma 24 as $\lim_{X\to\overline{X}} U(X) = \lim_{X\to\overline{X}} V(X) = g^*(\overline{X},0)$. Therefore, consider the case where $\overline{X}$ is not an absorbing state ($\overline{m} \neq 0$) and without loss of generality suppose $\lim_{X\to\overline{X}} V(X) > \lim_{X\to\overline{X}} U(X)$. From Lemma 24, $\lim_{X\to\overline{X}} V(X) - U(X) = \overline{m}(\overline{v}' - \overline{u}')/r$. Therefore, given $\overline{m} < 0$, this implies $\overline{v}' < \overline{u}'$. By continuity, there exists an $X^* \in (\underline{X},\overline{X})$ such that $V(X^*) > U(X^*)$ and $V'(X^*) < U'(X^*)$. From the proof of Lemma 7, this implies that $V(X) > U(X)$ and $V'(X) < U'(X)$ for all $X \in (\underline{X},X^*)$. This implies $V(X) - U(X)$ is decreasing in $X$ for $X \in (\underline{X},X^*)$, and therefore, $\lim_{X\to\underline{X}} V(X) - U(X) > V(X^*) - U(X^*) > 0$. Therefore, $\underline{m}(\underline{v}' - \underline{u}')/r > 0$. Given $\underline{m} > 0$, this implies $\overline{v}' > \overline{u}'$ which is a contradiction. Therefore, $\lim_{X\to\overline{X}} V(X) = \lim_{X\to\overline{X}} U(X)$. The case of $p = \underline{X}$ is analogous □

### D.2 Additional results for boundary conditions when $\mathcal{X} = \mathbb{R}$

The following additional results are used to establish the boundary conditions outlined in Appendix A.4 when $\mathcal{X} = \mathbb{R}$ and $g$ is bounded.

**Lemma 26.** *Suppose $\mathcal{X} = \mathbb{R}$ and $g$ is bounded. If $U$ is a bounded solution of Eq. (7), then there exists a $\delta > 0$ such that for $|X| > \delta$, $U$ is monotone and for $p \in \{-\infty, \infty\}$, $U_p \equiv \lim_{X \to p} U(X)$ exists and $\lim_{X \to p} U'(X) = 0$.*

*Proof.* Suppose $U$ is a bounded solution of Eq. (7) and it is not monotone near $p = \infty$. Then for all $\delta > 0$, there exists an increasing sequence $(X_n)_{n \in \mathbb{N}}$ of alternating consecutive local maxima and local minima of $U$, where $X_1 > \delta$. Thus $U'(X_n) = 0$ and $U''(X_n) \leq 0$ for the maxima and $U'(X_n) = 0$ and $U''(X_n) \geq 0$ for the minima. Given Eq. (6), a static Nash equilibrium is played at any $X$ such that $U'(X) = 0$, yielding flow payoff $g^*(X, 0)$. From Eq. (7), this implies $g^*(X_n, 0) \geq U(X_n)$ in the case of a maximum and $g^*(X_n, 0) \leq U(X_n)$ in the case of a minimum. Thus, the oscillation of $g^*(X, 0)$ on $[\delta, \infty)$ is at least as large as the oscillation of $U$, and therefore, $g^*(X, 0)$ is not monotone for large $X$. But by Assumption 3, $\psi'(X, z)$ is monotone for $X > \delta_0$, and therefore, $\psi(X, z)$ is also monotone for sufficiently large $X$. Therefore, $\psi(X, 0) = g^*(X, 0)$ is also monotone for sufficiently large $X$. This is a contradiction. Thus, there exists a $\delta$ such that for $X > \delta$, $U$ is monotone. The existence of $\lim_{X \to \infty} U(X)$ follows from $U$ bounded and monotone for large $X$.

Next suppose $\liminf_{X \to \infty} U'(X) \neq \limsup_{X \to \infty} U'(X)$. Then for all $\delta > 0$, by the continuity of $U'$, there exists a $z$ and an increasing sequence $(X_n)_{n \in \mathbb{N}}$ of alternating consecutive $X$ such that $X_1 > \delta$, $U'(X_n) = z$ and $U''(X_n) \leq 0$ for $n$ odd, $U'(X_n) = z$ and $U''(X_n) \geq 0$ for $n$ even, with one inequality for $U''$ strict. From Eq. (7), this implies $U(X_n) \leq \psi(X_n, z)$ for $n$ odd and $\psi(X_n, z) \leq U(X_n)$ for $n$ even, with one inequality strict. Thus, the oscillation of $\psi(X, z)$ is at least as large as the oscillation of $U$. But by Assumption 3, $\psi'(X, z)$ is monotone for $X > \delta_0$, and therefore, $\psi(X, z)$ is also monotone for sufficiently large $X$. Therefore, it must be that $\liminf_{X \to \infty} U'(X) = \limsup_{X \to \infty} U'(X)$. Let $U'_\infty$ denote this limit. Given $U$ is bounded, it must be that $U'_\infty = 0$. The case of $p = -\infty$ is analogous. $\square$

**Lemma 27.** *Suppose $\mathcal{X} = \mathbb{R}$, $g$ is bounded and $f : \mathbb{R} \to \mathbb{R}$ has linear growth or slower. Then any bounded solution $U$ of Eq. (7) satisfies*

$$\begin{aligned} \liminf_{X \to p} |f(X)| U'(X) &\leq 0 \leq \limsup_{X \to p} |f(X)| U'(X) \\ \liminf_{X \to p} f(X)^2 U''(X) &\leq 0 \leq \limsup_{X \to p} f(X)^2 U''(X) \end{aligned}$$

*for $p \in \{-\infty, \infty\}$.*

*Proof.* Consider $p = \infty$ and suppose $\liminf_{X\to\infty} |f(X)|U'(X) > 0$. Then there exists a $\delta_1 > 0$ and $\varepsilon_1 > 0$ such that when $X > \delta_1$, $|f(X)|U'(X) > \varepsilon_1$. By linear growth, there exists a $\delta_2 > 0$ and $M > 0$ such that when $X > \delta_2$, $|f(X)| \leq MX$. Take $\delta \equiv \max\{\delta_1, \delta_2\}$. Then when $X > \delta$, $U'(X) > \varepsilon_1/|f(X)| \geq \varepsilon_1/MX$. The antiderivative of $\varepsilon_1/MX$ is $(\varepsilon_1/M)\ln X$, which converges to $\infty$ as $X \to \infty$. This contradicts the boundedness of $U$. Therefore it must be that $\liminf_{X\to\infty} |f(X)|U'(X) \leq 0$. The proof is analogous to show $\limsup_{X\to\infty} |f(X)|U'(X) \geq 0$.

By Lemma 26, $U$ is monotone for large $X$. Without loss of generality, let $U$ be monotonically increasing. This implies $U'(X) \geq 0$ for sufficiently large $X$. Suppose $\liminf_{X\to\infty} f(X)^2U''(X) > 0$. Then there exists a $\delta_1 > 0$ and $\varepsilon > 0$ such that when $X > \delta_1$, $f(X)^2U''(X) > \varepsilon$. By linear growth, there exists a $\delta_2 > 0$ and $M > 0$ such that when $X > \delta_2$, $f(X)^2 \leq MX^2$. Take $\delta \equiv \max\{\delta_1, \delta_2\}$. Then when $X > \delta$, $U''(X) > \varepsilon/f(X)^2 \geq (\varepsilon/M)X^{-2} > 0$ and $U'(X)$ is strictly monotonically increasing. By Lemma 26, $U'(X) \to 0$. In order to have $U'(X) \to 0$ and $U'(X)$ strictly monotonically increasing, it must be that $U'(X) < 0$ for $X > \delta$. This is a contradiction, as $U'(X) \geq 0$ for sufficiently large $X$. Therefore, $\liminf_{X\to\infty} f(X)^2U''(X) \leq 0$. Suppose $\limsup_{X\to\infty} f(X)^2U''(X) < 0$. By similar reasoning, there exists a $\delta > 0$, $\varepsilon > 0$ and $M > 0$ such that when $X > \delta$, $U''(X) < -\varepsilon/f(X)^2 \leq (-\varepsilon/M)X^{-2} < 0$ and $U'(X)$ is strictly monotonically decreasing. Therefore, $\int_X^\infty U''(t)dt < \int_X^\infty (-\varepsilon/M)t^{-2}dt$ which implies $U'(X) > \varepsilon/MX$ since $U'(X) \to 0$. The antiderivative of $\varepsilon/MX$ is $(\varepsilon/M)\ln X$, which converges to $\infty$ as $X \to \infty$. This contradicts the boundedness of $U$. Therefore, $\limsup_{X\to\infty} f(X)^2U''(X) \geq 0$. The proof is analogous for the case of $p = -\infty$. □

### D.3 Alternative boundary conditions for $\mathcal{X} = \mathbb{R}$ and bounded $g$.

When $\mathcal{X} = \mathbb{R}$ and $g$ is bounded, Assumption 5 outlines an alternative sufficient condition for uniqueness.

**Assumption 5.** *Given $\mathcal{X} = \mathbb{R}$, there exists a $\delta_0 > 0$ such that for $|X| > \delta_0$, $|\mu^*(X, z)|/\sigma^*(X, z) \leq K$ for some $K > 0$.*

Theorem 6 establishes uniqueness when Assumption 5 holds.

**Theorem 6.** *Suppose $\mathcal{X} = \mathbb{R}$ and $g$ bounded. Assume Assumptions 1 to 3 and 5. For each initial state $X_0 \in \mathcal{X}$, there exists a unique PPE, which is Markov and characterized by the unique bounded solution $U$ of Eq. (7) on $\mathcal{X}$. The continuation value converges to the static Nash equilibrium payoff and intertemporal incentives collapse as the state grows large, $\lim_{X\to x}(U(X) - g^*(X, 0)) = 0$ and $\lim_{X\to x} \mu^*(X, U'(X))U'(X) = 0$ for $x \in \{-\infty, \infty\}$.*

Together with Lemmas 26 to 27, Lemmas 28 to 29 below establish the following boundary conditions for the case of $\mathcal{X} = \mathbb{R}$ and $g$ is bounded under Assumptions 1 to 3 and 5: any bounded solution $U$ of Eq. (7) satisfies $\lim_{X\to p} U(X) = g_p$, $\lim_{X\to p} \mu^*(X, U'(X))U'(X) = 0$, and $\lim_{X\to p} \sigma^*(X, U'(X))^2 U''(X) = 0$ for $p \in \{-\infty, \infty\}$, where $g_p \equiv \lim_{X\to p} g^*(X, 0)$. Note $g_p$ exists, given that $g$ is bounded and $g^*(\cdot, 0)$ is monotone for large $|X|$. The proof of Theorem 6 follows immediately from these boundary conditions and Steps (2) and (3) from Appendix A.4.

**Lemma 28.** *Suppose $\mathcal{X} = \mathbb{R}$ and $g$ is bounded. Any bounded solution $U$ of Eq. (7) satisfies $\lim_{X\to p} \mu^*(X, U'(X))U'(X) = 0$ for $p \in \{-\infty, \infty\}$.*

*Proof.* Consider $p = \infty$. Let $f(X)$ be a positive Lipschitz continuous function such that there exists a $K_1, K_2, \delta_1 > 0$ such that $|\mu^*(X, z)| \leq K_1 f(X)$ and $\sigma^*(X, z) \geq K_2 f(X)$ for $X > \delta_1$ and $z \in \mathbb{R}$. Such a function exists for $\mu^*$ given that it is Lipschitz continuous and bounded in $z$, and by Assumption 5, such a function exists that also satisfies the property for $\sigma^*$. I first show $\lim_{X\to\infty} f(X)U'(X) = 0$ for all $z \in \mathbb{R}$. Suppose $\limsup_{X\to\infty} |f(X)|U'(X) > 0$. By Lemma 27, $\liminf_{X\to\infty} |f(X)|U'(X) \leq 0$ since $f(X)$ is Lipschitz continuous, and therefore, has linear growth. Then there exist constants $K > k > 0$ such that $|f(X)|U'(X)$ crosses $k$ and $K$ infinitely many times as $X$ approaches $\infty$. Additionally, there exist $\delta > 0$ and $L > 0$ such that for $X > \delta$ with $|f(X)|U'(X) \in (k, K)$,

$$|U''(X)| = \left| \frac{2r(U(X) - g^*(X, U'(X))) - 2\mu^*(X, U'(X))U'(X)}{\sigma^*(X, U'(X))^2} \right| \leq \frac{L}{f(X)^2}.$$

where the equality follows from Eq. (7) and the inequality follows from the triangle inequality, $U$ and $g^*$ bounded, $\mu^*$ bounded by $f$, and Assumption 5. Given $f$ is Lipschitz continuous, there exists an $L_2 > 0$ such that $|f'(X)| < L_2$. Then for $X > \delta$ such that $|f(X)|U'(X) \in (k, K)$,

$$\begin{aligned} |(f(X)U'(X))'| &\leq |f'(X)U'(X)| + |f(X)U''(X)| = \left( |f'(X)| + \frac{|f(X)^2 U''(X)|}{|f(X)U'(X)|} \right) |U'(X)| \\ &\leq (L_2 + L/k)|U'(X)|, \end{aligned}$$

and therefore, $|U'(X)| \geq |(f(X)U'(X))'|/(L_2 + L/k)$. Therefore, the total variation of $U$ is at least $(K - k)/(L_2 + L/k) > 0$ on any interval where $|f(X)|U'(X)$ crosses $k$ and stays in $(k, K)$ until crossing $K$. This happens infinitely often in a neighborhood of $\infty$, which, given that $U$ is monotone for large $X$ by Lemma 26, implies that $U$ does not converge as $X \to \infty$. This is a contradiction by Lemma 26. Thus, $\limsup_{X\to\infty} |f(X)|U'(X) = 0$. By similar logic, $\liminf_{X\to\infty} |f(X)|U'(X) = 0$, and therefore, $\lim_{X\to\infty} |f(X)|U'(X) = 0$. Given $|\mu^*(X, U'(X))| \leq K_1 f(X)$, this implies $|\mu^*(X, U'(X))U'(X)| \leq K_1 |f(X)U'(X)| \to 0$ as $X \to \infty$. The case of $p = -\infty$ is analogous. $\square$

**Lemma 29.** *Suppose $\mathcal{X} = \mathbb{R}$ and $g$ is bounded. Let $U$ be a bounded solution of Eq. (7). Then for $p \in \{-\infty, \infty\}$, $\lim_{X\to p} U(X) = g_p$ and $\lim_{X\to p} \sigma^*(X, U'(X))^2 U''(X) = 0$.*

*Proof.* Consider $p = \infty$ and suppose $\lim_{X\to\infty} U(X) = U_\infty > g_\infty$. By Lemma 28, $\mu^*(X, U'(X))U'(X) \to 0$ as $X \to \infty$. Moreover, $g^*$ Lipschitz continuous, $U'(X) \to 0$ by Lemma 28, and $g^*(X, 0) \to g_\infty$ imply $g^*(X, U'(X)) \to g_\infty$ as $X \to \infty$. Plugging these limits into Eq. (7),

$$\begin{aligned}\lim_{X\to\infty} \sigma^*(X, U'(X))^2 U''(X) &= \lim_{X\to\infty} 2r(U(X) - g^*(X, U'(X))) - 2\mu^*(X, U'(X))U'(X) \\ &= 2r(U_\infty - g_\infty) > 0. \end{aligned} \tag{38}$$

By the Lipschitz continuity of $\sigma^*$ and $U'(X) \to 0$ from Lemma 28, there exists an $\delta, M > 0$ such that for $X > \delta$, $\sigma^*(X, U'(X)) \leq MX$. This implies

$$\lim_{X\to\infty} \sigma^*(X, U'(X))^2 U''(X) \leq \lim \inf_{X\to\infty} M^2 X^2 U''(X) \leq 0.$$

where the last equality follows from Lemma 27. This is a contradiction, since by Eq. (38) $\lim_{X\to\infty} \sigma^*(X, U'(X))^2 U''(X) > 0$. A similar contradiction holds for $U_\infty < g_\infty$. Therefore, $U_\infty = g_\infty$ and $\lim_{X\to\infty} \sigma^*(X, U'(X))^2 U''(X) = 0$ follows immediately. The case of $p = -\infty$ is analogous. □

# E Additional Material from Section 6

## E.1 Section 6.2

This model satisfies the assumptions in Section 3. Volatility is positive, except at the boundary states (Assumption 1). The state space is bounded. Therefore, the board's flow payoff is also bounded (Assumption 2.i). From Lemma 1, given current state $X$ and incentive weight $r\beta$, the board chooses intervention $a(X, r\beta) = \max\{-1, \min\{\beta X(2-X)/2c, 1\}\}$. The board will choose an intervention that increases the state when the equilibrium incentive weight is positive, and otherwise chooses an intervention that decreases the p. The sequentially rational action profile $(a(X, r\beta), \bar{b}(X, r\beta))$ is single-valued and Lipschitz continuous (Assumption 3), where $\bar{b}(X, r\beta) = \lambda a(X, r\beta)^2 + 1 - (1-X)^2$.

From Theorem 1, any solution $U$ to the optimality equation with equilibrium actions $a(X, U'(X)) = \max\{-1, \min\{U'(X)X(2-X)/2cr, 1\}\}$ and $\bar{b}(X, U'(X)) = \lambda a(X, U'(X))^2 + 1 - (1-X)^2$ characterizes a Markov equilibrium, where

$$U(X) = r(\bar{b}(a(X), X) - ca(X)^2) + X(2-X)((a(X) + \theta(d-X))U'(X) + U''(X)/2). \tag{39}$$

### E.2 Section 6.3

It is straightforward to compute the sequentially rational action profile for any $(z, X)$. Given incentive weight $z$, the government chooses an investment level to solve

$$\max_{a \in [0,\overline{a}]} -\frac{1}{2}a^2 + \frac{\theta_2 z}{r} a.$$

This results in sequentially rational investment level

$$a(X,z) = \begin{cases} \theta_2 z/r & \text{if } z/r \in [0, \overline{a}/\theta_2] \\ \overline{a} & \text{if } z/r > \overline{a}/\theta_2 \\ 0 & \text{if } z < 0 \end{cases}$$

for the government. When an innovator believes that the government will choose investment level $\tilde{a}$ and the current stock of intellectual capital is $X$, the innovator's best response is to select investment $\tilde{a}X/c$ if $\tilde{a}/c \leq \gamma$, and otherwise to choose the maximum possible investment, $\gamma X$. To reduce the number of cases, assume that an interior solution is always feasible for the innovator, $\overline{a} \leq \gamma c$. This results in sequentially rational investment level

$$b(X,z) = \begin{cases} \frac{\theta_2 z}{cr} X & \text{if } z/r \in [0, \overline{a}/\theta_2] \\ \frac{\overline{a}}{c} X & \text{if } z/r > \overline{a}/\theta_2 \\ 0 & \text{if } z < 0 \end{cases}$$

for each innovator. Note that $a(X,z)$ and $b(X,z)$ are unique for each $(X,z)$ and Lipschitz continuous (Assumption 3).

I first search for equilibria in which the government's optimal investment is an interior solution. Conjecture that there exists an equilibrium in which the continuation value is linear in the current level of intellectual capital. This means that $U'(X)$ is constant with respect to $X$ and $U''(X) = 0$. Taking the derivative of the optimality equation, Eq. (7), such an equilibrium must satisfy

$$rU'(X) = r\frac{dg^*}{dX} + U'(X)\frac{d\mu^*}{dX}. \tag{40}$$

Rearranging terms and plugging in the derivatives of Eq. (14) and Eq. (15), any solution to

$$z = \frac{\alpha\theta_2 z}{cr - \theta_1\theta_2 z/r + c\theta_3} \tag{41}$$

with $z/r \in [0, \overline{a}/\theta_2]$ is a candidate equilibrium slope. It is straightforward to verify that $z^* = 0$ is a solution. In an equilibrium with slope $z^* = 0$, neither the government nor the innovators invest, $a(X) = b(X) = 0$ for all $X$, and the government's equilibrium payoff is $U(X) = 0$. Due to the strategic complementarity, if the government doesn't invest, then neither will the innovators, yielding a payoff of zero for all players. If $\alpha = 0$ or $\theta_2 = 0$, $z^* = 0$ is also the unique solution, and therefore, the unique equilibrium. Intuitively, if the government doesn't receive a return on the innovators' investment or its own investment does not contribute to building intellectual capital, then it has no incentive to undertake costly investment.

There are also non-trivial equilibria that sustain positive investment. The unique non-zero solution to Eq. (41) is

$$\frac{z^*}{r} = \frac{cr - \alpha\theta_2 + c\theta_3}{\theta_1\theta_2}. \tag{42}$$

In order for this to be a valid solution, it must satisfy $z^*/r \in [0, \overline{a}/\theta_2]$. Recall that by assumption, $\overline{a} \leq \gamma c$ and $\gamma < (r + \theta_3)/\theta_1$. Allowing $\overline{a}$ and $\gamma$ to be as large as possible, subject to these constraints, yields $\overline{a}/\theta_2 \approx (cr + c\theta_3)/\theta_1\theta_2$ as the upper bound for $z^*/r$. It is clear from Eq. (42) that $z^*/r < (cr + c\theta_3)/\theta_1\theta_2$ for all $\alpha > 0$ and $\theta_2 > 0$. For the lower bound, $z^*/r > 0$ for all $r$ when $c\theta_3 > \alpha\theta_2$. Therefore, there exists an equilibrium that sustains positive investment and has slope Eq. (42) when $\gamma \approx (r + \theta_3)/\theta_1$, $\overline{a} = \gamma c$ and $c\theta_3 > \alpha\theta_2$. This equilibrium has non-zero equilibrium investment levels,

$$\begin{aligned} a(X) &= \frac{cr - \alpha\theta_2 + c\theta_3}{\theta_1} \\ b(X) &= \left(\frac{cr - \alpha\theta_2 + c\theta_3}{c\theta_1}\right) X, \end{aligned}$$

and continuation value[4]

$$U(X) = r\left(\frac{cr - \alpha\theta_2 + c\theta_3}{\theta_1\theta_2}\right) X + \frac{(cr - \alpha\theta_2 + c\theta_3)^2}{2\theta_1^2}.$$

To complete the characterization of Markov equilibria, it remains to determine whether there are equilibria with slopes $z < 0$ or $z/r > \overline{a}/\theta_2$. If there is an equilibrium with slope

[4]Given process $dX_t = \theta(M - X_t)dt + \sigma dZ_t$, the ergodic distribution of $X$ has mean $M$. From $\mu(a, b, X) = \theta_1 b(X) + \theta_2 a(X) - \theta_3 X$, $\theta = -\theta_1 b(X)/X + \theta_3 = \alpha\theta_2/c - r$. and

$$M = \frac{\theta_2 a(X)}{\theta} = \frac{c\theta_2(cr - \alpha\theta_2 + c\theta_3)}{\theta_1(\alpha\theta_2 - cr)} = \frac{c^2\theta_2\theta_3}{\theta_1(\alpha\theta_2 - cr)} - \frac{\theta_2 c}{\theta_1}.$$

Therefore, the equilibrium ergodic distribution of intellectual capital has mean $c^2\theta_2\theta_3/\theta_1(\alpha\theta_2 - cr) - c\theta_2/\theta_1$. As $r \to 0$, $M \to c^2\theta_3/\alpha\theta_1 - c\theta_2/\theta_1 = ca(X)/\alpha$. The expected flow payoff is

$$E[g(a(X), b(X), X)] = E[\alpha b(X) - a(X)^2/2] = \alpha a(X)E[X]/c - a(X)^2/2.$$

As $r \to 0$, $E[g(a(X), b(X), X)] \to (c\theta_3 - \alpha\theta_2)^2/2\theta_1^2$, which is equal to $\lim_{r\to 0} U(X)$ derived above.

$z < 0$, then from sequential rationality, $a(X, z) = b(X, z) = 0$, which leads to $U(X) = 0$. But then $U'(X) = 0$, which contradicts $z < 0$. Therefore, there are no Markov equilibria with slope $z < 0$. There may be a Markov equilibrium with slope $z/r > \overline{a}/\theta_2$. In such an equilibrium, from sequential rationality, $a(X, z) = \overline{a}$ and $b(X, z) = \overline{a}X/c$. Computing $g^*(X, z)$ and $\mu^*(X, z)$ for this case and plugging the derivatives into Eq. (40), the equilibrium slope must satisfy

$$\frac{z^*}{r} = \frac{\alpha\overline{a}}{cr - \theta_1\overline{a} + c\theta_3}. \tag{43}$$

and $z^*/r > \overline{a}/\theta_2$. These conditions are simultaneously satisfied when $\alpha > (cr - \theta_1\overline{a} + c\theta_3)/\theta_2$—therefore, there is an equilibrium with slope Eq. (43) and equilibrium investment levels $a(X) = \overline{a}$ and $b(X) = \overline{a}X/c$.